1 Homotopy

Definition (Chain complex). A *chain complex* is a sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

$d_i \circ d_{i+1} = 0$

Definition (Cochain complex). A *cochain complex* is a sequence of abelian groups and homomorphisms

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \longrightarrow \cdots$$

$$H_i(C_\cdot) = \frac{\ker(d_i : C_i \to C_{i-1})}{\operatorname{im}(d_{i+1} : C_{i+1} \to C_i)}. \qquad H^i(C^\cdot) = \frac{\ker(d^i : C^i \to C^{i+1})}{\operatorname{im}(d^{i-1} : C^{i-1} \to C^i)}.$$

$d^i \circ d^{i-1} = 0$

2 Singular (co)homology

Definition (Singular chain complex). We let $C_n(X)$ be the free abelian group on the set of singular n-simplices in X. More explicitly, we have

$$C_n(X) = \left\{ \sum n_\sigma \sigma : \sigma : \Delta^n \to X, n_\sigma \in \mathbb{Z}, \text{only finitely many } n_\sigma \text{ non-zero} \right\}.$$

We define $d_n : C_n(X) \to C_{n-1}(X)$ by

$$\sigma \mapsto \sum_{i=0}^{n} (-1)^i \sigma \circ \delta_i,$$

3 Four major tools of (co)homology

Theorem (Homotopy invariance theorem). Let $f \simeq g : X \to Y$ be homotopic maps. Then they induce the same maps on (co)homology, i.e.

Corollary. If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_\cdot(X) \to H_\cdot(Y)$ and $f^* : H^\cdot(Y) \to H^\cdot(X)$ are isomorphisms. (g_* and g^* are the inverses).

Theorem (Mayer-Vietoris theorem). Let $X = A \cup B$ be the union of two open subsets. We have inclusions

$$\begin{tikzcd} A \cap B \arrow[r, hook, "i_A"] \arrow[d, hook, "i_B"] & A \arrow[d, hook, "j_A"] \\ B \arrow[r, hook, "j_B"] & X \end{tikzcd}.$$

Then there are homomorphisms $\partial_{MV} : H_n(X) \to H_{n-1}(A \cap B)$ such that the following sequence is exact:

$$\xrightarrow{\partial_{MV}} H_n(A \cap B) \xrightarrow{i_{A*} \oplus i_{B*}} H_n(A) \oplus H_n(B) \xrightarrow{j_{A*} - j_{B*}} H_n(X) \xrightarrow{\partial_{MV}} H_{n-1}(A \cap B) \xrightarrow{i_{A*} \oplus i_{B*}} H_{n-1}(A) \oplus H_{n-1}(B) \xrightarrow{j_{A*} - j_{B*}} H_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_0(A) \oplus H_0(B) \xrightarrow{j_{A*} - j_{B*}} H_0(X) \longrightarrow 0$$

Furthermore, the Mayer-Vietoris sequence is *natural*, i.e. if $f : X = A \cup B \to Y = U \cup V$ satisfies $f(A) \subseteq U$ and $f(B) \subseteq V$, then the diagram

$$\begin{tikzcd} H_{n+1}(X) \arrow[r, "\partial_{MV}"] \arrow[d, "f_*"] & H_n(A \cap B) \arrow[r, "i_{A*} \oplus i_{B*}"] \arrow[d, "f|_{A \cap B*}"] & H_n(A) \oplus H_n(B) \arrow[r, "j_{A*} - j_{B*}"] \arrow[d, "f|_{A*} \oplus f|_{B*}"] & H_n(X) \arrow[d, "f_*"] \\ H_{n+1}(Y) \arrow[r, "\partial_{MV}"] & H_n(U \cap V) \arrow[r, "i_{U*} \oplus i_{V*}"] & H_n(U) \oplus H_n(V) \arrow[r, "j_{U*} - j_{V*}"] & H_n(Y) \end{tikzcd}$$

commutes.

For certain elements of $H_n(X)$, we can easily specify what ∂_{MV} does to it. The meat of the proof is to show that every element of $H_n(X)$ can be made to look like that. If $[a + b] \in H_n(X)$ is such that $a \in C_n(A)$ and $b \in C_n(B)$, then the map ∂_{MV} is specified by

$$\partial_{MV}([a + b]) = [d_n(a)] = [-d_n(b)] \in H_{n-1}(A \cap B).$$

Definition (Relative homology). Let $A \subseteq X$. We write $i : A \to X$ for the inclusion map. Then the map $i_n : C_n(A) \to C_n(X)$ is injective as well, and we write

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}.$$

The differential $d_n : C_n(X) \to C_{n-1}(X)$ restricts to a map $C_n(A) \to C_{n-1}(A)$, and thus gives a well-defined differential $d_n : C_n(X, A) \to C_{n-1}(X, A)$, sending $[c] \mapsto [d_n(c)]$. The *relative homology* is given by

$$H_n(X, A) = H_n(C_\cdot(X, A)).$$

We think of this as chains in X where we ignore everything that happens in A.

Theorem (Exact sequence for relative homology). There are homomorphisms $\partial : H_n(X, A) \to H_{n-1}(A)$ given by mapping

$$[[c]] \mapsto [d_n c].$$

This makes sense because if $c \in C_n(X)$, then $[c] \in C_n(X)/C_n(A)$. We know $[d_n c] = 0 \in C_{n-1}(X)/C_{n-1}(A)$. So $d_n c \in C_{n-1}(A)$. So this notation makes sense.

Moreover, there is a long exact sequence

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{q_*} H_{n-1}(X, A) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_0(X) \xrightarrow{q_*} H_0(X, A) \longrightarrow 0$$

where i_* is induced by $i : C_\cdot(A) \to C_\cdot(X)$ and q_* is induced by the quotient $q : C_\cdot(X) \to C_\cdot(X, A)$.

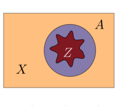
Definition (Map of pairs). Let (X, A) and (Y, B) be topological spaces with $A \subseteq X$ and $B \subseteq Y$. A *map of pairs* is a map $f : X \to Y$ such that $f(A) \subseteq B$.

Such a map induces a map $f_* : H_n(X, A) \to H_n(Y, B)$, and the exact sequence for relative homology is natural for such maps.

Theorem (Excision theorem). Let (X, A) be a pair of spaces, and $Z \subseteq A$ be such that $\bar{Z} \subseteq \mathring{A}$ (the closure is taken in X). Then the map

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$

is an isomorphism.



While we've only been talking about homology, everything so far holds analogously for cohomology too. It is again homotopy invariant, and there is a Mayer-Vietoris sequence (with maps $\partial_{MV} : H^n(A \cap B) \to H^{n+1}(X)$). The relative cohomology is defined by $C^\cdot(X, A) = \operatorname{Hom}(C_\cdot(X, A), \mathbb{Z})$ and so $H^*(X, A)$ is the cohomology of that. Similarly, excision holds.

Definition (Degree of a map). Let $f : S^n \to S^n$ be a map. The *degree* $\deg(f)$ is the unique integer such that under the identification $H_n(S^n) \cong \mathbb{Z}$, the map f_* is given by multiplication by $\deg(f)$.

Proposition.

(i) $\deg(\mathrm{id}_{S^n}) = 1$.

(ii) If f is not surjective, then $\deg(f) = 0$.

(iii) We have $\deg(f \circ g) = (\deg f)(\deg g)$.

(iv) Homotopic maps have equal degrees.

Lemma. Let M be a d-dimensional manifold (i.e. a Hausdorff, second-countable space locally homeomorphic to $\mathbb{R}^d$). Then

$$H_n(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & n = d \\ 0 & \text{otherwise} \end{cases}.$$

This is known as the *local homology*.

Theorem (Snake lemma). Suppose we have a short exact sequence of complexes

$$0 \longrightarrow A_\cdot \xrightarrow{i_\cdot} B_\cdot \xrightarrow{q_\cdot} C_\cdot \longrightarrow 0 .$$

Then there are maps

$$\partial : H_n(C_\cdot) \to H_{n-1}(A_\cdot)$$

such that there is a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{q_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{q_*} H_{n-1}(C) \longrightarrow \cdots.$$

Lemma (Five lemma). Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\ell"] & B \arrow[r, "g"] \arrow[d, "m"] & C \arrow[r, "h"] \arrow[d, "n"] & D \arrow[r, "j"] \arrow[d, "p"] & E \arrow[d, "q"] \\ A' \arrow[r, "r"] & B' \arrow[r, "s"] & C' \arrow[r, "t"] & D' \arrow[r, "u"] & E' \end{tikzcd}$$

If the two rows are exact, m and p are isomorphisms, q is injective and ℓ is surjective, then n is also an isomorphism.

Corollary. Let $f : (X, A) \to (Y, B)$ be a map of pairs, and that any two of $f_* : H_*(X, A) \to H_*(Y, B)$, $H_*(X) \to H_*(Y)$ and $H_*(A) \to H_*(B)$ are isomorphisms. Then the third is also an isomorphism.

Definition (Chain homotopy). A *chain homotopy* between chain maps $f_\cdot, g_\cdot : C_\cdot \to D_\cdot$ is a collection of homomorphisms $F_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d^D_{n+1} \circ F_n + F_{n-1} \circ d^C_n : C_n \to D_n$$

for all n.

Lemma. If $f_\cdot$ and $g_\cdot$ are chain homotopic, then $f_* = g_* : H_*(C_\cdot) \to H_*(D_\cdot)$.

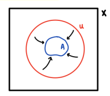
4 Reduced homology

Definition (Reduced homology). Let X be a space, and $x_0 \in X$ a basepoint. We define the *reduced homology* to be $\tilde{H}_*(X) = H_*(X, \{x_0\})$.

LES of a pair $\Rightarrow$ $\tilde{H}_n(X) \cong H_n(X)$ $\forall n \geq 1$

Definition (Good pair). We say a pair (X, A) is *good* if there is an open set U containing $\bar{A}$ such that the inclusion $A \hookrightarrow U$ is a deformation retract, i.e. there exists a homotopy $H : [0, 1] \times U \to U$ such that

$$H(0, x) = x$$
$$H(1, x) \in A$$
$$H(t, a) = a \text{ for all } a \in A, t \in [0, 1].$$



Theorem. If (X, A) is good, then the natural map

$$H_*(X, A) \longrightarrow H_*(X/A, \underbrace{A/A}_{\text{collapses to a point.}}) = \tilde{H}_*(X/A)$$

is an isomorphism.

5 Cell complexes

Lemma. If $A \subseteq X$ is a subcomplex, then the pair (X, A) is *good*.

Lemma. Let X be a cell complex. Then

(i)
$$H_i(X^n, X^{n-1}) = \begin{cases} 0 & i \neq n \\ \bigoplus_{\alpha \in I_n} \mathbb{Z} & i = n \end{cases}.$$

(ii) $H_i(X^n) = 0$ for all $i > n$.

(iii) $H_i(X^n) \to H_i(X)$ is an isomorphism for $i < n$.

For a cell complex X, let

$$C^{\text{cell}}_n(X) = H_n(X^n, X^{n-1}) \cong \bigoplus_{\alpha \in I_n} \mathbb{Z}.$$

We define $d^{\text{cell}}_n : C^{\text{cell}}_n(X) \to C^{\text{cell}}_{n-1}(X)$ by the composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{q} H_{n-1}(X^{n-1}, X^{n-2}) .$$

Theorem. $H^{\text{cell}}_n(X) \cong H_n(X)$.

Corollary. If X is a finite cell complex, then $H_n(X)$ is a finitely-generated abelian group for all n, generated by at most $|I_n|$ elements. In particular, if there are no n-cells, then $H_n(X)$ vanishes.

Definition (Cellular cohomology). We define *cellular cohomology* by

$$C^n_{\text{cell}}(X) = H^n(X^n, X^{n-1})$$

and let d^n_{cell} be the composition

$$H^n(X^n, X^{n-1}) \xrightarrow{q^*} H^n(X^n) \xrightarrow{\partial} H^{n+1}(X^{n+1}, X^n).$$

$$C^\cdot_{\text{cell}}(X) \cong \operatorname{Hom}(C^{\text{cell}}_\cdot(X), \mathbb{Z}).$$

6 (Co)homology with coefficients

Definition ((Co)homology with coefficients). Let A be an abelian group, and X be a topological space. We let

$$C_\cdot(X; A) = C_\cdot(X) \otimes_{\mathbb{Z}} A$$

with differentials $d \otimes \mathrm{id}_A$. In other words $C_\cdot(X; A)$ is the abelian group obtained by taking the direct sum of many copies of A, one for each singular simplex.

$C^\cdot(X; A) = \operatorname{Hom}(C_\cdot(X), A)$.

We call A the "coefficients", since a general member of $C_\cdot(X; A)$ looks like

$$\sum n_\sigma s, \quad \text{where } n_\sigma \in A, \quad \sigma : \Delta^n \to X.$$

7 Euler characteristic

Definition (Euler characteristic). Let X be a cell complex. We let

$$\chi(X) = \sum_n (-1)^n \text{ number of } n\text{-cells of } X \in \mathbb{Z}.$$

$$\chi_{\mathbb{Z}}(X) = \sum_n (-1)^n \operatorname{rank} H_n(X; \mathbb{Z}).$$

$$\chi_{\mathbb{F}}(X) = \sum_n (-1)^n \dim_{\mathbb{F}} H_n(X; \mathbb{F}).$$

If $0 \to A \to B \to C \to 0$ is exact, then

rank(B) = rank(A) + rank(C) by 1^{st} iso thm.

8 Cup product

Definition (Cup product). Let R be a commutative ring, and $\phi \in C^k(X; R)$, $\psi \in C^\ell(X; R)$. Then $\phi \smile \psi \in C^{k+\ell}(X; R)$ is given by

$$(\phi \smile \psi)(\sigma : \Delta^{k+\ell} \to X) = \phi(\sigma|_{[v_0, \ldots, v_k]}) \overset{\text{ring multiplication}}{\cdot} \psi(\sigma|_{[v_k, \ldots, v_{k+\ell}]}). \quad \in R$$

$$H^*(X; R) = \bigoplus_{n \geq 0} H^n(X; R).$$

Lemma. If $\phi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$, then

$$d(\phi \smile \psi) = (d\phi) \smile \psi + (-1)^k \phi \smile (d\psi).$$

Corollary. The cup product induces a well-defined map

$$\begin{aligned} \smile : H^k(X; R) \times H^\ell(X; R) &\longrightarrow H^{k+\ell}(X; R) \\ ([\phi], [\psi]) &\longmapsto [\phi \smile \psi] \end{aligned}$$

Proposition. $(H^*(X; R), \smile, [1])$ is a unital ring.

Proposition. Let R be a commutative ring. If $[\alpha] \in H^k(X; R)$ and $[\beta] \in H^\ell(X; R)$, then we have

$$[\alpha] \smile [\beta] = (-1)^{k\ell} [\beta] \smile [\alpha]$$

Proposition. The cup product is natural, i.e. if $f : X \to Y$ is a map, and $\alpha, \beta \in H^*(Y; R)$, then

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

Definition (Cross product). Let $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ be the projection maps. Then we have a *cross product*

$$\begin{aligned} \times : H^k(X; R) \otimes_R H^\ell(Y; R) &\longrightarrow H^{k+\ell}(X \times Y; R) \\ a \otimes b &\longmapsto (\pi_X^* a) \smile (\pi_Y^* b) \end{aligned}.$$

Note that the diagonal map $\Delta : X \to X \times X$ given by $\Delta(x) = (x, x)$ satisfies

$$\Delta^*(a \times b) = a \smile b$$

for all $a, b \in H^*(X; R)$. So these two products determine each other.

There is also a *relative cup product*

$$\smile : H^k(X, A; R) \otimes H^\ell(X; R) \to H^{k+\ell}(X, A; R)$$

9 Künneth theorem and universal coefficients theorem

Theorem (Künneth's theorem). Let R be a commutative ring, and suppose that $H^n(Y; R)$ is a free R-module for each n. Then the cross product map

$$\bigoplus_{k+\ell=n} H^k(X; R) \otimes H^\ell(Y; R) \xrightarrow{\times} H^n(X \times Y; R)$$

is an isomorphism for every n, for every finite cell complex X.

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$$

is an isomorphism of graded rings.

$$a \otimes_R b \longmapsto a \times b$$

where we define multiplication in LHS graded ring to be

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} (a \smile c) \otimes (b \smile d)$$

Theorem (Universal coefficients theorem for (co)homology). Let R be a $\underbrace{\text{PID}}_{\text{principal ideal domain (i.e. } \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{Z}_p)}$ and M an R-module. Then there is a natural map

$$H_*(X; R) \otimes M \to H_*(X; M).$$

If $H_*(X; R)$ is a free module for each n, then this is an isomorphism. Similarly, there is a natural map

$$H^*(X; M) \to \operatorname{Hom}_R(H_*(X; R), M), ,$$

which is an isomorphism again if $H^*(X; R)$ is free.

Homology of $X \times Y$: can define chain complexes $C_n(X \times Y) = \bigoplus_{k+\ell=n} C_k(X) \otimes C_\ell(Y)$

and the differential is $d(u \otimes v) = du \otimes v + (-1)^k u \otimes dv$.

10 Vector bundles

10.1 Vector bundles

Definition (Vector bundle). Let X be a space. A (real) *vector bundle* of dimension d over X is a map $\pi : E \to X$, with a (real) vector space structure on each *fiber* $E_x = \pi^{-1}(\{x\})$, subject to the local triviality condition: for each $x \in X$, there is a neighbourhood U of x and a homeomorphism $\varphi : E|_U = \pi^{-1}(U) \to U \times \mathbb{R}^d$ such that the following diagram commutes

$$\begin{tikzcd} E|_U \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^d \arrow[d, "\pi_1"', "\text{pr}_1"] \\ & U \end{tikzcd},$$

and for each $y \in U$, the restriction $\varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^d$ is a *linear* isomorphism for each $y \in U$. This maps is known as a *local trivialization*.

Definition (Pullback of vector bundles). Let $\pi : E \to X$ be a vector bundle, and $f : Y \to X$ a map. We define the *pullback*

$$f^*E = \{(y, e) \in Y \times E : f(y) = \pi(e)\}.$$

This has a map $f^*\pi : f^*E \to Y$ given by projecting to the first coordinate. The vector space structure on each fiber is given by the identification $(f^*E)_y = E_{f(y)}$.

Example (Grassmannian manifold). We let

$$X = \mathrm{Gr}_k(\mathbb{R}^n) = \{k\text{-dimensional linear } \overset{\text{(vector subspaces)}}{\text{subgroups}} \text{ of } \mathbb{R}^n\}.$$

$E = \{(V, v) \in \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n : v \in V\}$.

$U = \{W \in \mathrm{Gr}_k(\mathbb{R}^n) : W \cap V^\perp = \{0\}\}$. call this bundle $\gamma^{\mathbb{R}}_{k,n} \to \mathrm{Gr}_k(\mathbb{R}^n)$.

The *normal bundle* of M in N is

$$\nu_{M \subseteq N} = \frac{i^*TN}{TM}.$$

Theorem (Tubular neighbourhood theorem). Let $M \subseteq N$ be a smooth submanifold. Then there is an open neighbourhood U of M and a homeomorphism $\nu_{M\subseteq N} \to U$, and moreover, this homeomorphism is the identity on M (where we view M as a submanifold of $\nu_{M\subseteq N}$ by the image of the zero section).

Definition (Partition of unity). Let X be a compact Hausdorff space, and $\{U_\alpha\}_{\alpha\in I}$ be an open cover. A *partition of unity subordinate to* $\{U_\alpha\}$ is a collection of functions $\lambda_\alpha : X \to [0, \infty)$ such that

(i) $\mathrm{supp}(\lambda_\alpha) = \overline{\{x \in X : \lambda_\alpha(x) > 0\}} \subseteq U_\alpha$.

(ii) Each $x \in X$ lies in finitely many of these $\mathrm{supp}(\lambda_\alpha)$. really need an open nhood of X that touches finitely many $\mathrm{supp}(\lambda_\alpha)$.

(iii) For each x, we have

$$\sum_{\alpha\in I} \lambda_\alpha(x) = 1.$$

finitely many $\Rightarrow$ sum is well defined.

Lemma. Let $\pi : E \to X$ be a vector bundle over a compact Hausdorff space. Then there is some N such that E is a vector subbundle of $X \times \mathbb{R}^N$.

Theorem. There is a correspondence

$$\left\{\begin{matrix}\text{homotopy classes}\\ \text{of maps}\\ f : X \to \mathrm{Gr}_d(\mathbb{R}^\infty)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}d\text{-dimensional}\\ \text{vector bundles}\\ \pi : E \to X\end{matrix}\right\}$$

$$[f] \longmapsto f^*\gamma_d^{\mathbb{R}}$$

$$[f_\pi] \longleftarrow \pi$$

10.2 Vector bundle orientations

$E^\# = E \setminus s_0(X)$. $\quad H^i(E_x, E_x^\#; R) \cong H^i(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}; R) = \begin{cases} R & i = d \\ 0 & \text{otherwise}\end{cases}$

Definition (R-orientation). A *local R-orientation* of E at $x \in X$ is a choice of R-module generator $\varepsilon_x \in H^d(E_x, E_x^\#; R)$.

An *R-orientation* is a choice of local R-orientation $\{\varepsilon_x\}_{x\in X}$ which are compatible in the following way: if $U \subseteq X$ is open on which E is trivial, and $x, y \in U$, then under the homeomorphisms (and in fact linear isomorphisms):

$$\begin{tikzcd} E_x \arrow[dr, hook] \arrow[drrr, bend left, "h_x"] & & & \\ & E|_U \arrow[r, "\varphi_\alpha", "\cong"'] & U \times \mathbb{R}^d \arrow[r, "\pi_2"] & \mathbb{R}^d \\ E_y \arrow[ur, hook] \arrow[urrr, bend right, "h_y"'] & & & \end{tikzcd}$$

i.e. ε_x and ε_y correspond to the same elt of $H^d(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\}; R)$

the map

$$h_y^* \circ (h_x^{-1})^* : H^d(E_x, E_x^\#; R) \to H^d(E_y, E_y^\#; R)$$

sends ε_x to ε_y. Note that this definition does not depend on the choice of φ_U, because we used it twice, and they cancel out.

Lemma. If $\{U_\alpha\}_{\alpha\in I}$ is a family of covers such that for each $\alpha, \beta \in I$, the homeomorphism

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^d \xleftarrow[\cong]{\varphi_\alpha} E|_{U_\alpha \cap U_\beta} \xrightarrow[\varphi_\beta]{\cong} (U_\alpha \cap U_\beta) \times \mathbb{R}^d$$

gives an orientation preserving map from $(U_\alpha \cap U_\beta) \times \mathbb{R}^d$ to itself, i.e. has a positive determinant on each fiber, then E is orientable for any R.

10.3 The Thom isomorphism theorem

Theorem (Thom isomorphism theorem). Let $\pi : E \to X$ be a d-dimensional vector bundle, and $\{\varepsilon_x\}_{x\in X}$ be an R-orientation of E. Then

(i) $H^i(E, E^\#; R) = 0$ for $i < d$.

(ii) There is a unique class $u_E \in H^d(E, E^\#; R)$ which restricts to ε_x on each fiber. This is known as the *Thom class*.

(iii) The map Φ given by the composition

$$H^i(X; R) \xrightarrow[\cong]{\pi^*} H^i(E; R) \xrightarrow{-\smile u_E} H^{i+d}(E, E^\#; R)$$

is an isomorphism.

remember: absolute $\smile$ relative = relative

Note that (i) follows from (iii), since $H^i(X; R) = 0$ for $i < 0$.

Definition (Euler class). Let $\pi : E \to X$ be a vector bundle. We define the *Euler class* $e(E) \in H^d(X; R)$ by the image of u_E under the composition

$$u_E \in H^d(E, E^\#; R) \xrightarrow[q^*]{} H^d(E; R) \xrightarrow[\cong]{s_0^*} H^d(X; R)\ . \mapsto e(E)$$

Theorem. We have

$$u_E \smile u_E = \Phi(e(E)) = \pi^*(e(E)) \smile u_E \in H^*(E, E^\#; R).$$

Lemma. If $\pi : E \to X$ is a d-dimensional R-module vector bundle with d odd, then $2e(E) = 0 \in H^d(X; R)$.

Theorem. If there is a section $s : X \to E$ which is nowhere zero, then $e(E) = 0 \in H^d(X; R)$.

10.4 Gysin sequence

Definition (Sphere bundle). Let $\pi : E \to X$ be a vector bundle, and let $\langle\cdot,\cdot\rangle : E \otimes E \to \mathbb{R}$ be an inner product, and let

$$S(E) = \{v \in E; \langle v, v\rangle = 1\} \subseteq E.$$

This is the *sphere bundle* associated to E.

$$\begin{tikzcd} H^{i+d}(E, E^\#) \arrow[r, "q^*"] & H^{i+d}(E) \arrow[r, "i^*"] \arrow[d, shift left, "s_0^*"] & H^{i+d}(E^\#) \arrow[r, "\partial"] \arrow[d, "j^*", "\cong"'] & H^{i+d+1}(E, E^\#) \\ H^i(X) \arrow[u, "\Phi"] \arrow[r, "\cdot\smile e(E)"] \arrow[ur, phantom, "(1)"] & H^{i+d}(X) \arrow[r, "p^*"', "\text{by dfn}"] \arrow[u, shift left, "\pi^*"] & H^{i+d}(S(E)) \arrow[r, "p_!"', "\text{define it to be composition.}"] & H^{i+1}(X) \arrow[u, "\Phi"'] \end{tikzcd}$$

11 Manifolds and Poincaré duality

11.1 Compactly supported cohomology

Definition (Compactly-supported cohomology). The *compactly supported cohomology* of X is

$$H_c^*(X) = H^*(C_c^\bullet(X)).$$

Note that we can write

$$C_c^\bullet(X) = \bigcup_{K \text{ compact}} C^\bullet(X, X \setminus K) \subseteq C^\bullet(X).$$

$$H_c^*(X) \cong \varinjlim_{\mathbf{K}(X)} H^n(X, X \setminus K).$$

Extension by zero: $K \subseteq U \subseteq X$, K compact, U open. Then excision gives us an iso: $H^*(U, U\setminus K; R) \cong H^*(X, X\setminus K; R)$ and we get a map taking the direct limit called extension by zero: $i_* : H_c^*(U) \to H_c^*(X)$.

We call this "extension by zero". Indeed, this is how the cohomology class works — if you have a cohomology class ϕ on U supported on $K \subseteq U$, then given any simplex in X, if it lies inside U, we know how to evaluate it. If it lies outside K, then we just send it to zero. Then by barycentric subdivision, we can assume every simplex is either inside U or outside K, so we are done.

Lemma. We have

$$H_c^i(\mathbb{R}^d; R) \cong \begin{cases} R & i = d \\ 0 & \text{otherwise}\end{cases}.$$

$H^n(X \mid K; R) = H^n(X, X \setminus K; R)$.

Proposition. Let $K, L \subseteq X$ be compact. Then there is a long exact sequence

$$H^n(X \mid K \cap L) \xrightarrow{\text{inc}-\text{inc}} H^n(X \mid K) \oplus H^n(X \mid L) \xrightarrow{\text{inc}+\text{inc}} H^n(X \mid K \cup L) \xrightarrow{\partial} H^{n+1}(X \mid K \cap L) \to H^{n+1}(X \mid K) \oplus H^{n+1}(X \mid L) \longrightarrow \cdots,$$

Corollary. Let X be a manifold, and $X = A \cup B$, where A, B are open sets. Then there is a long exact sequence

$$H_c^n(A \cap B) \xrightarrow{(i_A)_* \oplus (i_B)_*} H_c^n(A) \oplus H_c^n(B) \xrightarrow{(j_A)_* - (j_B)_*} H_c^n(X) \xrightarrow{\partial} H_c^{n+1}(A \cap B) \longrightarrow H_c^{n+1}(A) \oplus H_c^{n+1}(B) \longrightarrow \cdots$$

11.2 Orientation of manifolds

Definition (Local R-orientation of manifold). Fr a d-manifold M, a local R-orientation of M at x is an R-module generator $\mu_x = H_d(M \mid x; R)$.

Definition (R-orientation). An *R-orientation* of M is a collection $\{\mu_x\}_{x\in M}$ of local R-orientations such that if

$$\varphi : \mathbb{R}^d \xrightarrow{\sim} U \subseteq M$$

is a chart of M, and $p, q \in \mathbb{R}^d$, then the composition of isomorphisms if $p, q \in D^d$

$$\begin{tikzcd} H_d(M \mid \varphi(p)) \arrow[r, "\sim", "\text{excision}"'] & H_d(U \mid \varphi(p)) & H_d(\mathbb{R}^d \mid p) \arrow[l, "\sim", "\varphi_*"'] \arrow[d, "\sim"] & H_d(\mathbb{R}^d \mid D^d) \arrow[l, "\sim"'] \arrow[d, equal] \\ H_d(M \mid \varphi(q)) \arrow[r, "\sim", "\text{excision}"'] & H_d(U \mid \varphi(q)) & H_d(\mathbb{R}^d \mid q) \arrow[l, "\sim", "\varphi_*"'] & H_d(\mathbb{R}^d \mid D^d) \arrow[l, "\sim"'] \end{tikzcd}$$

sends μ_x to μ_y, where the vertical isomorphism is induced by a translation of $\mathbb{R}^d$.

Definition (Orientation-preserving homeomorphism). For a ~~homomorphism~~ homeomorphism $f : U \to V$ with $U, V \in \mathbb{R}^d$ open, we say f is R-orientation-preserving if for each $x \in U$, and $y = f(x)$, the composition

$$\begin{tikzcd} H_d(\mathbb{R}^d \mid 0; R) \arrow[r, "\text{translation}"] & H_d(\mathbb{R}^d \mid x; R) \arrow[r, "\text{excision}"] & H_d(U \mid x; R) \arrow[d, "f_*"] \\ H_d(\mathbb{R}^d \mid 0; R) \arrow[r, "\text{translation}"] & H_d(\mathbb{R}^d \mid y; R) \arrow[r, "\text{excision}"] & H_d(V \mid y; R) \end{tikzcd}$$

is the identity $H_d(\mathbb{R}^d \mid 0; R) \to H_d(\mathbb{R}^d \mid 0; R)$.

Lemma.

(i) If $R = \mathbb{F}_2$, then every manifold is R-orientable.

(ii) If $\{\varphi_\alpha : \mathbb{R}^d \to U_\alpha \subseteq M\}$ is an open cover of M by Euclidean space such that each homeomorphism

$$\mathbb{R}^d \supseteq \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \xleftarrow{\varphi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\varphi_\beta^{-1}} \varphi_\beta^{-1}(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^d$$

is orientation-preserving, then M is R-orientable.

Theorem. Let M be an R-oriented manifold and $A \subseteq M$ be compact. Then

(i) There is a unique class $\mu_A \in H_d(M \mid A; R)$ which restricts to $\mu_x \in H_d(M \mid x; R)$ for all $x \in A$.

(ii) $H_i(M \mid A; R) = 0$ for $i > d$.

Definition (Fundamental class). The *fundamental class* of an R-oriented manifold is the unique class $[M]$ that restricts to μ_x for each $x \in M$.

11.3 Poincaré duality

Theorem (Poincaré duality). Let M be a d-dimensional R-oriented manifold. Then there is a map

$$D_M : H_c^k(M; R) \to H_{d-k}(M; R)$$

that is an isomorphism.

Definition (Cap product). The *cap product* is defined by

$\cdot \frown \cdot : C^k(X; R) \otimes C_\ell(X; R) \to C_{\ell-k}(X; R)$

$\phi \frown \sigma := \sigma|_{[v_0, \ldots, v_{\ell-k}]} \cdot \phi(\sigma|_{[v_{\ell-k}, \ldots, v_\ell]})$

So that $\psi(\phi \frown \sigma) = (\psi \smile \phi)(\sigma)$

$d(\phi \frown \sigma) = (-1)^{\ell-k} d\phi \frown \sigma + \phi \frown d\sigma$.

Lemma. If $f : X \to Y$ is a map, and $x \in H_k(X; R)$ And $y \in H^\ell(Y; R)$, then we have

$f_*(f^*(\phi) \frown x) = \phi \frown f_*(x)$

Corollary. For any compact d-dimensional R-oriented manifold M, the map

$$[M] \frown \cdot : \underset{\text{ordinary cohomology}}{H^\ell(M; R)} \to H_{d-\ell}(M; R)$$

is an isomorphism.

Corollary. Let M be an odd-dimensional compact manifold. Then the Euler characteristic $\chi(M) = 0$.

Theorem. Let M be a d-dimensional compact R-oriented manifold, and consider the following pairing:

$$\langle\cdot,\cdot\rangle : H^k(M; R) \otimes H^{d-k}(M, R) \longrightarrow R$$

$$[\varphi] \otimes [\psi] \longmapsto (\varphi \smile \psi)[M]$$

If $H_*(M; R)$ is free, then $\langle\cdot,\cdot\rangle$ is non-singular

11.4 Applications

Definition (Signature of manifold). Let M be a $4k$-dimensional $\mathbb{Z}$-oriented manifold. Then the signature is the number of positive eigenvalues of

$$\langle\cdot,\cdot\rangle : H^{2k}(M; \mathbb{R}) \otimes H^{2k}(M; \mathbb{R}) \to \mathbb{R}$$

minus the number of negative eigenvalues. We write this as $\mathrm{sgn}(M)$.

Fact. If $M = \partial W$ for some compact $4k + 1$-dimensional manifold W with boundary, then $\mathrm{sgn}(M) = 0$.

Definition (Degree of map). If M, N are d-dimensional compact connected $\mathbb{Z}$-oriented manifolds, and $f : M \to N$, then

$$f_*([M]) \in H_d(N, \mathbb{Z}) \cong \mathbb{Z}[N].$$

So $f_*([M]) = k[N]$ for some k. This k is called the *degree* of f, written $\deg(f)$.

11.5 Intersection product

Write $\nu_{N\subseteq M}$ for the normal bundle of N in M. Picking a metric on TM, we can decompose

$$i^*TM \cong TN \oplus \nu_{N\subseteq M},$$

we have an isomorphism

$$H^i(\nu_{N\subseteq M}, \nu^\#_{N\subseteq M}; R) \cong H_{d-i}(N; R).$$

Theorem. The Poincaré dual of a submanifold is (the extension by zero of) the normal Thom class.

The P.D of $[N] \in H_n(M;R)$ is the extension by zero of the Thom class of $\nu_{N\subseteq M}$.

Idea: under the iso $H^{d-n}(\nu, \nu^\#; R) \cong H_c^{d-n}(U; R) \cong H_d(N; R)$, $[N]$ corresponds to some $\varepsilon_{N\subseteq M}$. Looks like and smells like a Thom class, $\varepsilon_{N\subseteq M}$ is a Thom class for normal bundle.

Both $[N]$ and $\varepsilon_{N\subseteq M}$ get mapped to $D_M^{-1}([N])$ under the respective maps

$$\begin{tikzcd} H_n(N; R) \arrow[r, "\sim"] \arrow[d, "\wr"] & H_n(U; R) & H_c^{d-n}(U; R) \arrow[l, "\cong"'] \arrow[r, "\text{ext by 0}"] & H^{d-n}(M; R) \\ H^{d-n}(\nu, \nu^\#; R) \arrow[r] & H_c^{d-n}(U; R) \arrow[r, "\cong"] & H_c^{d-n}(U; R) \arrow[r, "\text{ext. by 0}"] & H^{d-n}(M; R). \end{tikzcd}$$

Whenever the intersection is transverse, the intersection $N \cap W$ will be a submanifold of M, and of N and W as well. Moreover,

$$(\nu_{N\cap W\subseteq M})_x = (\nu_{N\subseteq M})_x \oplus (\nu_{W\subseteq M})_x.$$

Now consider the inclusions

$$i_N : N \cap W \hookrightarrow N$$
$$i_W : N \cap W \hookrightarrow W.$$

Then we have

$$\nu_{N\cap W\subseteq M} = i_N^*(\nu_{N\subseteq M}) \oplus i_W^*(\nu_{W\subseteq M}).$$

So with some abuse of notation, we can write

$$i_N^*\mathcal{E}_{N\subseteq M} \smile i_W^*\mathcal{E}_{W\subseteq M} \in H^*(\nu_{N\cap W\subseteq M}, \nu^\#_{N\cap W\subseteq M}; R),$$

and we can check this gives the Thom class. So we have

$$D_M^{-1}([N]) \smile D_M^{-1}([W]) = D_M^{-1}([N \cap W]).$$

Definition (Intersection product). The *intersection product* on the homology of a compact manifold is given by

$$H_{n-k}(M) \otimes H_{n-\ell}(M) \longrightarrow H_{n-k-\ell}(M)$$

$$(a, b) \longmapsto a \cdot b = D_M(D_M^{-1}(a) \smile D_M^{-1}(b))$$